

Gaussian states minimize the output entropy of one-mode quantum Gaussian channels

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Abstract

We prove that Gaussian thermal input states minimize the output von Neumann entropy of any one-mode phase-covariant quantum Gaussian channel among all the input states with a given entropy. The starting point of the proof is the recent result stating that Gaussian thermal input states saturate the $p \rightarrow q$ norms of one-mode quantum-limited Gaussian channels. Quantum Gaussian channels model in the quantum regime the attenuation and the noise that affect any electromagnetic signal. This result is crucial to prove the converse theorems for both the triple trade-off region and the capacity region for broadcast communication of the Gaussian quantum-limited amplifier.

1 Introduction

Signal attenuation and noise unavoidably affect electromagnetic communications through metal wires, optical fibers or free space. Since the energy carried by an electromagnetic pulse is quantized, quantum effects must be taken into account [1]. They become relevant for low-intensity signals, such

as for satellite communications, where the receiver can be reached by only few photons for each bit of information [2]. In the quantum regime, signal attenuation and noise are modeled by quantum Gaussian channels [3–7].

The maximum achievable communication rate of a channel depends on the minimum noise achievable at its output, that is quantified by the output von Neumann entropy [5, 8]. We prove the longstanding conjecture [9–14] stating that Gaussian thermal input states minimize the output entropy of one-mode phase-covariant quantum Gaussian channels for fixed input entropy. The best lower bound to the constrained minimum output entropy known so far follows from the quantum Entropy Power Inequality [15–17]. However, this inequality is not saturated by Gaussian states, and it is thus not sufficient to prove the conjecture. The constrained minimum output entropy conjecture has recently been proven for the one-mode quantum-limited attenuator [18, 19] directly attacking the problem with Lagrange multipliers. Unfortunately the same proof does not work in the presence of noise or amplification. In this paper we prove the conjecture in a completely different way, starting from the recent result stating that Gaussian input states saturate the $p \rightarrow q$ norms of one-mode quantum-limited Gaussian channels.

Our result is necessary to prove the converse theorems that guarantee the optimality of Gaussian encodings for two communication tasks involving the quantum-limited amplifier [20]. The first is the triple trade-off coding, where a sender wants to transmit both classical and quantum information and generate shared entanglement, or he wants to transmit both public and private classical information and to generate a shared secret key. The second is broadcast communication, where a sender wants to transmit classical information to two receivers.

2 Setup

2.1 Gaussian quantum systems

We consider the Hilbert space of one harmonic oscillator, or one mode of electromagnetic radiation. Its ladder operator \hat{a} satisfies the canonical commutation relation $[\hat{a}, \hat{a}^\dagger] = \hat{\mathbb{I}}$, and its Hamiltonian $\hat{N} = \hat{a}^\dagger \hat{a}$ counts the number of excitations, or photons. The vector annihilated by \hat{a} is the vacuum

$|0\rangle$, from which the Fock states are built:

$$|n\rangle = \frac{(\hat{a}^\dagger)^n}{\sqrt{n!}}|0\rangle, \quad \langle m|n\rangle = \delta_{mn}, \quad \hat{N}|n\rangle = n|n\rangle. \quad (2.1)$$

An operator diagonal in the Fock basis is called Fock-diagonal.

2.2 Quantum Gaussian states

A quantum Gaussian state corresponds to a geometric probability distribution for the energy:

$$\hat{\omega}_E = \sum_{n=0}^{\infty} \frac{1}{E+1} \left(\frac{E}{E+1} \right)^n |n\rangle\langle n|, \quad E \geq 0, \quad \text{Tr} [\hat{N} \hat{\omega}_E] = E. \quad (2.2)$$

Its von Neumann entropy is

$$S(\hat{\omega}_E) = (E+1) \ln(E+1) - E \ln E := g(E). \quad (2.3)$$

2.3 Quantum attenuator

The quantum attenuator $\mathcal{E}_{\lambda,E}$ of transmissivity $0 \leq \lambda \leq 1$ mixes the input state $\hat{\rho}$ with the thermal Gaussian state $\hat{\omega}_E$ of an ancillary quantum system B through a beamsplitter of transmissivity λ . The beamsplitter is implemented by the unitary operator

$$\hat{U}_\lambda = \exp \left(\left(\hat{a}^\dagger \hat{b} - \hat{a} \hat{b}^\dagger \right) \arccos \sqrt{\lambda} \right), \quad (2.4)$$

that satisfies

$$\hat{U}_\lambda^\dagger \hat{a} \hat{U}_\lambda = \sqrt{\lambda} \hat{a} + \sqrt{1-\lambda} \hat{b}, \quad (2.5)$$

where \hat{b} is the ladder operator of the ancilla system B (see Section 1.4.2 of [21]), and

$$\mathcal{E}_{\lambda,E}(\hat{\rho}) = \text{Tr}_B \left[\hat{U}_\lambda (\hat{\rho} \otimes \hat{\omega}_E) \hat{U}_\lambda^\dagger \right]. \quad (2.6)$$

For $E = 0$, the state of the environment is the vacuum and the attenuator is quantum-limited.

The quantum attenuator sends thermal Gaussian states into themselves [5]:

$$\mathcal{E}_{\lambda,E}(\hat{\omega}_{E'}) = \hat{\omega}_{\lambda E' + (1-\lambda)E} \quad \forall E' \geq 0. \quad (2.7)$$

2.4 Quantum amplifier

The quantum amplifier $\mathcal{A}_{\kappa,E}$ of amplification parameter $\kappa \geq 1$ performs a two-mode squeezing on the input state $\hat{\rho}$ and the thermal Gaussian state $\hat{\omega}_E$ of an ancillary quantum system B . The squeezing is implemented by the unitary operator

$$\hat{U}_\kappa = \exp \left(\left(\hat{a}^\dagger \hat{b}^\dagger - \hat{a} \hat{b} \right) \operatorname{arccosh} \sqrt{\kappa} \right) , \quad (2.8)$$

that satisfies

$$\hat{U}_\kappa^\dagger \hat{a} \hat{U}_\kappa = \sqrt{\kappa} \hat{a} + \sqrt{\kappa - 1} \hat{b}^\dagger , \quad (2.9)$$

where \hat{b} is the ladder operator of the ancilla system B (see Section 1.4.4 of [21]), and

$$\mathcal{A}_{\kappa,E}(\hat{\rho}) = \operatorname{Tr}_B \left[\hat{U}_\kappa (\hat{\rho} \otimes \hat{\omega}_E) \hat{U}_\kappa^\dagger \right] . \quad (2.10)$$

For $E = 0$, the state of the environment is the vacuum and the amplifier is quantum-limited.

Also the quantum amplifier sends thermal states into themselves [5]:

$$\mathcal{A}_{\kappa,E}(\hat{\omega}_{E'}) = \hat{\omega}_{\kappa E' + (\kappa - 1)(E + 1)} \quad \forall E' \geq 0 . \quad (2.11)$$

The quantum attenuators and amplifiers together constitute the phase-covariant quantum Gaussian channels. The quantum-limited attenuator and amplifier are the building blocks for this class of channels: any of them can be decomposed as [7, 22–24]:

$$\mathcal{E}_{\lambda,E} = \mathcal{A}_{\kappa'} \circ \mathcal{E}_{\lambda'} , \quad (2.12)$$

$$\mathcal{A}_{\kappa,E} = \mathcal{A}_{\kappa''} \circ \mathcal{E}_{\lambda''} , \quad (2.13)$$

where

$$\begin{aligned} \lambda' &= \frac{\lambda}{(1 - \lambda) E + 1} , & \kappa' &= (1 - \lambda) E + 1 , \\ \lambda'' &= \frac{1}{\left(1 - \frac{1}{\kappa}\right) E + 1} , & \kappa'' &= \kappa \left(\left(1 - \frac{1}{\kappa}\right) E + 1 \right) , \end{aligned} \quad (2.14)$$

and where we have defined for simplicity

$$\mathcal{E}_\lambda = \mathcal{E}_{\lambda,0} , \quad \mathcal{A}_\kappa = \mathcal{A}_{\kappa,0} . \quad (2.15)$$

3 Main results

We start recalling the main result of [18].

Theorem 1 ([18], Theorem 1). *Gaussian thermal input states minimize the output entropy of the quantum-limited attenuator among all the input states with a given entropy, i.e. for any input state $\hat{\rho}$ and any $0 \leq \lambda \leq 1$*

$$S(\mathcal{E}_\lambda(\hat{\rho})) \geq g(\lambda g^{-1}(S(\hat{\rho}))) . \quad (3.1)$$

In this paper we first extend Theorem 1 to the quantum-limited amplifier.

Theorem 2. *Gaussian thermal input states minimize the output entropy of the quantum-limited amplifier among all the input states with a given entropy, i.e. for any input state $\hat{\rho}$ and any $\kappa \geq 1$*

$$S(\mathcal{A}_\kappa(\hat{\rho})) \geq S(\mathcal{A}_\kappa(\hat{\omega})) = g(\kappa g^{-1}(S(\hat{\rho})) + \kappa - 1) . \quad (3.2)$$

Proof. See Section 4. □

Then, we use Theorems 1 and 2 together to extend the result to any one-mode phase-covariant Gaussian channel.

Theorem 3. *Gaussian thermal input states minimize the output entropy of any one-mode phase-covariant quantum Gaussian channel among all the input states with a given entropy, i.e. for any $0 \leq \lambda \leq 1$, $\kappa \geq 1$, $E \geq 0$ and any quantum state $\hat{\rho}$*

$$S(\mathcal{E}_{\lambda,E}(\hat{\rho})) \geq g(\lambda g^{-1}(S(\hat{\rho})) + (1 - \lambda)E) , \quad (3.3)$$

$$S(\mathcal{A}_{\kappa,E}(\hat{\rho})) \geq g(\kappa g^{-1}(S(\hat{\rho})) + (\kappa - 1)(E + 1)) . \quad (3.4)$$

Proof. With the help of (2.12) and Theorem 2 we have

$$S(\mathcal{E}_{\lambda,E}(\hat{\rho})) = S(\mathcal{A}_{\kappa'}(\mathcal{E}_{\lambda'}(\hat{\rho}))) \geq g(\kappa' g^{-1}(S(\mathcal{E}_{\lambda'}(\hat{\rho}))) + \kappa' - 1) . \quad (3.5)$$

Since g is increasing, also g^{-1} is increasing, and $S \mapsto g(\kappa' g^{-1}(S) + \kappa' - 1)$ is increasing, too. Then, Theorem 1 implies

$$S(\mathcal{E}_{\lambda,E}(\hat{\rho})) \geq g(\kappa' \lambda' g^{-1}(S(\hat{\rho})) + \kappa' - 1) , \quad (3.6)$$

i.e. the claim.

The proof for $\mathcal{A}_{\kappa,E}$ is identical. □

4 Proof of Theorem 2

Since \mathcal{A}_1 is the identity channel, the claim is trivial for $\kappa = 1$. We can then assume $\kappa > 1$. For $S(\hat{\rho}) = 0$ the claim is implied by the proof of the Gaussian minimum output entropy conjecture [7, 22, 23, 25], stating that the vacuum input state minimizes the output entropy of any phase-covariant quantum Gaussian channel among all the input states, and in particular among the input states with zero entropy. We can then assume $S(\hat{\rho}) > 0$.

The starting point of our proof is the recent result stating that for any $\kappa > 1$ and any $1 < p < q$ thermal Gaussian states saturate the $p \rightarrow q$ norm of the one-mode quantum-limited amplifier.

Theorem 4. *For any $1 < p < q$ and any $\kappa > 1$ the $p \rightarrow q$ norm of \mathcal{A}_κ is saturated by a thermal Gaussian state $\hat{\omega}$ (that depends on κ , p and q), i.e. for any operator \hat{X} with $\|\hat{X}\|_p < \infty$*

$$\frac{\|\mathcal{A}_\kappa(\hat{X})\|_q}{\|\hat{X}\|_p} \leq \frac{\|\mathcal{A}_\kappa(\hat{\omega})\|_q}{\|\hat{\omega}\|_p}. \quad (4.1)$$

Here

$$\|\hat{X}\|_p := \left(\text{Tr} \left(\hat{X}^\dagger \hat{X} \right)^{\frac{p}{2}} \right)^{\frac{1}{p}} \quad (4.2)$$

is the Schatten p norm [26, 27], defined as the l^p norm of the singular values of the operator.

Lemma 5. *For any $\kappa > 1$, any $S > 0$ and any $1 < q < 3/2$ there exists $1 < p < q$ such that the $p \rightarrow q$ norm of \mathcal{A}_κ is saturated by the thermal Gaussian state with entropy S .*

Proof. From Theorem 4 the $p \rightarrow q$ norm of \mathcal{A}_κ is saturated by some thermal Gaussian state. It is then sufficient to prove that we can choose p such that this state has entropy S . We use a different parametrization of thermal Gaussian states, with

$$z := \frac{E}{E+1}, \quad 0 \leq z < 1, \quad (4.3)$$

$$\hat{\omega}_z = \sum_{n=0}^{\infty} (1-z) z^n |n\rangle \langle n|. \quad (4.4)$$

The transformation rule for z following from (2.11) is $\mathcal{A}_\kappa(\hat{\omega}_z) = \hat{\omega}_{z'}$, with

$$z' = \frac{z + \kappa - 1}{\kappa} . \quad (4.5)$$

Let $0 < \bar{z} < 1$ be such that $\hat{\omega}_{\bar{z}}$ has entropy S . The claim follows if we prove that we can choose p such that the function

$$z \mapsto \frac{\|\mathcal{A}_\kappa(\hat{\omega}_z)\|_q}{\|\hat{\omega}_z\|_p} , \quad 0 < z < 1 \quad (4.6)$$

has a global maximum at $z = \bar{z}$. We can compute

$$\begin{aligned} \ln \|\hat{\omega}_z\|_p &= \ln(1 - z) - \frac{1}{p} \ln(1 - z^p) , \\ \frac{\partial}{\partial z} \ln \|\hat{\omega}_z\|_p &= -\frac{1 - z^{p-1}}{(1 - z)(1 - z^p)} . \end{aligned} \quad (4.7)$$

For any $0 < z < 1$, let z' be such that $\mathcal{A}_\kappa(\hat{\omega}_z) = \hat{\omega}_{z'}$. Similarly, let \bar{z}' be such that $\mathcal{A}_\kappa(\hat{\omega}_{\bar{z}}) = \hat{\omega}_{\bar{z}'}$. With the help of (4.5) we have

$$\frac{\partial}{\partial z} \ln \frac{\|\mathcal{A}_\kappa(\hat{\omega}_z)\|_q}{\|\hat{\omega}_z\|_p} = \frac{\phi(z, p) - \phi(z', q)}{1 - z} , \quad (4.8)$$

where

$$\phi(z, p) := \frac{1 - z^{p-1}}{1 - z^p} . \quad (4.9)$$

Let us first prove that we can choose $1 < p < q$ such that

$$\left. \frac{\partial}{\partial z} \ln \frac{\|\mathcal{A}_\kappa(\hat{\omega}_z)\|_q}{\|\hat{\omega}_z\|_p} \right|_{z=\bar{z}} = 0 , \quad (4.10)$$

i.e. $\phi(\bar{z}, p) = \phi(\bar{z}', q)$. The function $p \mapsto \phi(\bar{z}, p)$ is continuous for $p \geq 1$. The claim then follows if we prove that

$$\phi(\bar{z}, 1) < \phi(\bar{z}', q) < \phi(\bar{z}, q) . \quad (4.11)$$

The first inequality in (4.11) follows since $\phi(\bar{z}, 1) = 0 < \phi(\bar{z}', q)$. The second inequality in (4.11) follows since $\bar{z}' > \bar{z}$ and $\phi(z, q)$ is decreasing in z . This last property holds since

$$\frac{\partial}{\partial z} \ln \phi(z, p) = \frac{z^{p-2}}{1 - z^{p-1}} \frac{1 + p(z - 1) - z^p}{1 - z^p} < 0 . \quad (4.12)$$

There remains to prove that any \bar{z} satisfying (4.10) is a global maximizer for (4.6). For $z = 0$ we have $z' = 1 - 1/\kappa > 0$, hence

$$\phi(z', q) < \phi(0, q) = 1 = \phi(0, p) , \quad (4.13)$$

and

$$\left. \frac{\partial}{\partial z} \ln \frac{\|\mathcal{A}_\kappa(\hat{\omega}_z)\|_q}{\|\hat{\omega}_z\|_p} \right|_{z=0} > 0 . \quad (4.14)$$

Then, $z = 0$ cannot be a maximizer. Moreover,

$$\lim_{z \rightarrow 1} \phi(z, p) = 1 - \frac{1}{p} < 1 - \frac{1}{q} = \lim_{z \rightarrow 1} \phi(z, q) , \quad (4.15)$$

hence

$$\lim_{z \rightarrow 1} \frac{\partial}{\partial z} \ln \frac{\|\mathcal{A}_\kappa(\hat{\omega}_z)\|_q}{\|\hat{\omega}_z\|_p} = -\infty , \quad (4.16)$$

and the supremum cannot be achieved for $z \rightarrow 1$. The maximizer must then be in the open interval $(0, 1)$, and the claim follows if we prove that (4.8) vanishes only for $z = \bar{z}$. Equivalently, we will prove that $z \mapsto \phi(z, p)/\phi(z', q)$ is strictly decreasing. We have

$$\frac{\partial}{\partial z} \ln \frac{\phi(z, p)}{\phi(z', q)} = \frac{f(z', q) - f(z, p)}{1 - z} , \quad (4.17)$$

where

$$f(z, p) := z^{p-2} \frac{1-z}{1-z^{p-1}} \left(p \frac{1-z}{1-z^p} - 1 \right) . \quad (4.18)$$

We must then prove that

$$f(z', q) < f(z, p) . \quad (4.19)$$

We recall that $1 < p < q < 3/2$, hence $z \mapsto z^{p-1}$ is concave, and

$$\frac{\partial}{\partial z} \left(z^{p-2} \frac{1-z}{1-z^{p-1}} \right) = z^{p-1} \frac{(p-1)(1-z) + z^{p-1} - 1}{(z - z^p)^2} \leq 0 . \quad (4.20)$$

It follows that $z \mapsto z^{p-2} \frac{1-z}{1-z^{p-1}}$ is positive and decreasing. Since $z \mapsto z^p$ is convex,

$$z \mapsto \psi(z, p) := p \frac{1-z}{1-z^p} - 1 \quad (4.21)$$

is decreasing. Since $\lim_{z \rightarrow 1} \psi(z, p) = 0$, ψ is also positive. Then, $z \mapsto f(z, p)$ is decreasing. Since $z' > z$, we have

$$f(z', q) - f(z, p) \leq f(z, q) - f(z, p) . \quad (4.22)$$

Since $p < q$, the claim (4.19) follows if we prove that $p \mapsto f(z, p)$ is decreasing. We have

$$\begin{aligned} \frac{\partial}{\partial p} f(z, p) &= \frac{z^{p-2} (1-z)^2}{(1-z^p)^2 (1-z^{p-1})^2} \\ &\times \left((1-z^{p-1})(1-z^p) + (1-z^{2p-1}) \ln z^{p-1} - \frac{z \ln z}{1-z} (1-z^{p-1})^2 \right) . \end{aligned} \quad (4.23)$$

We need the following inequalities:

$$-\frac{z \ln z}{1-z} < \sqrt{z} , \quad (4.24)$$

$$\ln z^{p-1} < z^{p-1} - 1 - \frac{(1-z^{p-1})^2}{2} . \quad (4.25)$$

(4.24) follows from $t < \sinh t$ with $t = -\frac{\ln z}{2} > 0$. (4.25) follows from the Taylor series

$$\ln(1-x) = -\sum_{n=1}^{\infty} \frac{x^n}{n} \quad (4.26)$$

with $x = 1 - z^{p-1} > 0$. We then get

$$\frac{\partial}{\partial p} f(z, p) < -\frac{z^{p-2} (1-z)^2 \left(1 - z^{p-\frac{1}{2}}\right) \left(1 - 2z^{\frac{1}{2}} + z^{p-\frac{1}{2}}\right)}{2(1-z^p)^2} . \quad (4.27)$$

Since $p < 3/2$, we have

$$1 - 2z^{\frac{1}{2}} + z^{p-\frac{1}{2}} > (1 - \sqrt{z})^2 > 0 , \quad (4.28)$$

then $\frac{\partial}{\partial p} f(z, p) < 0$ and the claim follows. \square

Let $\hat{\rho}$ be a quantum state with $0 < S(\hat{\rho}) < \infty$, and let $\hat{\omega}$ be the thermal Gaussian state with $S(\hat{\rho}) = S(\hat{\omega})$. For $1 < q < 3/2$, let $1 < p(q) < q$ be the

exponent such that the $p(q) \rightarrow q$ norm of \mathcal{A}_κ is saturated by $\hat{\omega}$. Since the Rényi q -entropy is decreasing in q [7], we have for any $1 < q < 3/2$

$$\begin{aligned}
S(\mathcal{A}_\kappa(\hat{\rho})) &\geq \frac{q}{1-q} \ln \|\mathcal{A}_\kappa(\hat{\rho})\|_q \\
&\geq \frac{q}{1-q} \ln \frac{\|\mathcal{A}_\kappa(\hat{\omega})\|_q \|\hat{\rho}\|_{p(q)}}{\|\hat{\omega}\|_{p(q)}} \\
&= \frac{q}{1-q} \ln \|\mathcal{A}_\kappa(\hat{\omega})\|_q \\
&\quad + \frac{q}{p(q)} \frac{p(q)-1}{q-1} \left(\frac{p(q)}{1-p(q)} \ln \|\hat{\rho}\|_{p(q)} - \frac{p(q)}{1-p(q)} \ln \|\hat{\omega}\|_{p(q)} \right) .
\end{aligned} \tag{4.29}$$

We also have [7]

$$\lim_{p \rightarrow 1} \frac{p}{1-p} \ln \|\hat{\omega}\|_p = \lim_{p \rightarrow 1} \frac{p}{1-p} \ln \|\hat{\rho}\|_p = S(\hat{\rho}) , \tag{4.30}$$

and

$$\lim_{q \rightarrow 1} \frac{q}{1-q} \ln \|\mathcal{A}_\kappa(\hat{\omega})\|_q = S(\mathcal{A}_\kappa(\hat{\omega})) . \tag{4.31}$$

Since $1 < p(q) < q$, we have

$$\lim_{q \rightarrow 1} p(q) = 1 , \quad 0 \leq \frac{q}{p(q)} \frac{p(q)-1}{q-1} \leq 1 . \tag{4.32}$$

Then, the claim follows taking the limit $q \rightarrow 1$ in (4.29).

5 Conclusions

We have proved that Gaussian thermal input states minimize the output von Neumann entropy of any one-mode phase-covariant quantum Gaussian channel among all the input states with a given entropy. This result finally permits to prove the optimality of Gaussian encodings for both the triple trade-off coding and broadcast communication with the quantum-limited amplifier [20]. The future challenge is the extension of our result to the multimode scenario. However, our proof relies on Theorem 4, which relies on the majorization result of Ref. [19], which fails in the multimode scenario (see [28], Section IV.A).

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